

On the weak* continuity of $LUC(\mathcal{G})^*$ -module action on $LUC(\mathcal{X}, \mathcal{G})^*$ related to \mathcal{G} -space \mathcal{X}

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Abstract

Associated with a locally compact group \mathcal{G} and a \mathcal{G} -space \mathcal{X} there is a Banach subspace $LUC(\mathcal{X}, \mathcal{G})$ of $C_b(\mathcal{X})$, which has been introduced and studied by Lau and Chu in [4]. In this paper, we study some properties of the first dual space of $LUC(\mathcal{X}, \mathcal{G})$. In particular, we introduce a left action of $LUC(\mathcal{G})^*$ on $LUC(\mathcal{X}, \mathcal{G})^*$ to make it a Banach left module and then we investigate the Banach subalgebra $\mathfrak{Z}(\mathcal{X}, \mathcal{G})$ of $LUC(\mathcal{G})^*$, as the topological centre related to this module action, which contains $M(\mathcal{G})$ as a closed subalgebra. Also, we show that the faithfulness of this module action is related to the properties of the action of \mathcal{G} on \mathcal{X} and we extend the main results of Lau [14] from locally compact groups to \mathcal{G} -spaces. Sufficient and/or necessary conditions for the equality $\mathfrak{Z}(\mathcal{X}, \mathcal{G}) = M(\mathcal{G})$ or $LUC(\mathcal{G})^*$ are given. Finally, we apply our results to some special cases of \mathcal{G} and \mathcal{X} for obtaining various examples whose topological centres $\mathfrak{Z}(\mathcal{X}, \mathcal{G})$ are $M(\mathcal{G})$, $LUC(\mathcal{G})^*$ or neither of them.

Mathematics Subject Classification (2010). 46H25, 43A05, 43A10, 43A85.

Key words. \mathcal{G} -space, left uniformly continuous function, complex Radon measure, measure algebra, left module action, topological centre.

1 Introduction

Let \mathcal{G} be a locally compact group. Then the Banach space $LUC(\mathcal{G})^*$, the topological dual of the space of all bounded left uniformly continuous functions on \mathcal{G} , is a Banach algebra equipped with the Arens-type product. In fact, the product of m and n in $LUC(\mathcal{G})^*$ is defined by

$$\langle m \odot n, f \rangle = \langle m, n \diamond f \rangle, \quad (n \diamond f)(s) = \langle n, {}_s f \rangle, \quad {}_s f(t) = f(st), \quad (1)$$

where $f \in LUC(\mathcal{G})$ and $s, t \in \mathcal{G}$. In general, this product is not separately weak* to weak* continuous on $LUC(\mathcal{G})^*$, and in recent years there has been shown considerable interest by harmonic analysts in the characterization of the following space

$$\mathfrak{Z}(\mathcal{G}) = \left\{ m \in LUC(\mathcal{G})^* : n \mapsto m \odot n \text{ is weak* to weak* continuous} \right\}.$$

As far as we know the subject, the starting point of the study of the space $\mathfrak{Z}(\mathcal{G})$ is the paper by Zappa [19]. In details, Zappa proved that $\mathfrak{Z}(\mathbb{R})$ is precisely $M(\mathbb{R})$, where \mathbb{R} is the additive group of real number and $M(\mathbb{R})$ is the Banach space of all complex Radon measures on \mathbb{R} . This result was extended to all abelian locally compact groups by Grosser and Losert in [11], and to all locally compact groups by Lau in [14].

In this paper, considering \mathcal{X} as a locally compact Hausdorff space on which \mathcal{G} acts continuously from the left, following Lau and Chu [4] we introduce the Banach space $LUC(\mathcal{X}, \mathcal{G})$. Then, we present a left action of $LUC(\mathcal{G})^*$ on $LUC(\mathcal{X}, \mathcal{G})^*$, as an extension of the natural action of $M(\mathcal{G})$ on $M(\mathcal{X})$, to make a Banach left $LUC(\mathcal{G})^*$ -module. In particular, we investigate the faithfulness of the action of $M(\mathcal{G})$ (resp. $LUC(\mathcal{G})^*$) on $M(\mathcal{X})$ (resp. $LUC(\mathcal{X}, \mathcal{G})^*$) in relation to the action of \mathcal{G} on \mathcal{X} . Also, we prove that if \mathcal{X} is a transitive \mathcal{G} -space, then $M(\mathcal{X})$ is an $LUC(\mathcal{G})^*$ -submodule of $LUC(\mathcal{X}, \mathcal{G})^*$ just when \mathcal{X} is compact. Moreover, the main purpose of this work is to study the topological centre problem related to this module action. To this end, we introduce the Banach subspace $\mathfrak{Z}(\mathcal{X}, \mathcal{G})$ of $LUC(\mathcal{G})^*$, as the topological centre related to this module action, which contains $M(\mathcal{G})$ as a closed subalgebra. Furthermore, apart from some characterization of the space $\mathfrak{Z}(\mathcal{X}, \mathcal{G})$, we extend the main results of Lau [14] from locally compact groups to \mathcal{G} -spaces. Finally, we apply our results to some special cases of \mathcal{G} and \mathcal{X} to give examples with

- $\mathfrak{Z}(\mathcal{X}, \mathcal{G}) = M(\mathcal{G})$;
- $M(\mathcal{G}) \subsetneq \mathfrak{Z}(\mathcal{X}, \mathcal{G}) = LUC(\mathcal{G})^*$;
- $M(\mathcal{G}) \subsetneq \mathfrak{Z}(\mathcal{X}, \mathcal{G}) \subsetneq LUC(\mathcal{G})^*$.

The motivation of the study presented in this paper comes from some works of Lau and his coauthors [4, 12, 13, 14, 15] and some recent developments of the concept of regularity of bilinear maps were introduced and first studied by Arens [2]. It is known that the Arens regularity of module actions has been a major tool in the study of Banach algebras. For example, Arens regularity of module actions as a generalization of Arens regularity of Banach algebras were introduced and first studied by Filali and Eshaghi Gordji [6] and they used this notion to answer some questions regarding Arens regularity of Banach algebras raised by Lau and Ülger [16]. Also in [17], Arens regularity of module actions were considered by Mohammadzadeh and Vishki to investigate the conditions under which the second adjoint of a derivation into a dual Banach module is again a derivation, that extends the results of Dales, Rodriguez-Palacios and Velasco in [5] for a general derivation.

2 Some prerequisites

Throughout this paper, \mathcal{G} is a locally compact group with left Haar measure λ and identity element e , Δ refers to the modular function on \mathcal{G} and the notation $C_b(\mathcal{G})$ is used to denote the space of all bounded complex-valued continuous functions on \mathcal{G} with the supremum norm. Also, the subspaces $C_c(\mathcal{G})$ and $C_0(\mathcal{G})$ of $C_b(\mathcal{G})$ refer to the space of all functions with compact support and the space of all functions vanishing at infinity, respectively. Also, for all $s \in \mathcal{G}$ and $f \in C_b(\mathcal{G})$ we define ${}_s f$, the left translation of f by s , as ${}_s f(t) = f(st)$, $t \in \mathcal{G}$. A function f in $C_b(\mathcal{G})$ is called left uniformly continuous if the mapping $s \mapsto {}_s f$ from \mathcal{G} into $(C_b(\mathcal{G}), \|\cdot\|_\infty)$ is continuous. As usual, we mean by $LUC(\mathcal{G})$ the Banach space of all left uniformly continuous functions on \mathcal{G} . Moreover, the notation $M(\mathcal{G})$ is used to denote the measure algebra of \mathcal{G} consisting of all complex regular Borel measures on \mathcal{G} with the total variation norm and the convolution product “ $*$ ” defined by the formula

$$\langle \mu * \nu, f \rangle = \int_{\mathcal{G}} \int_{\mathcal{G}} f(st) d\mu(s) d\nu(t)$$

for all $\mu, \nu \in M(\mathcal{G})$ and $f \in C_0(\mathcal{G})$. It is well-known that $M(\mathcal{G})$ is the topological dual of $C_0(\mathcal{G})$ with the pairing

$$\langle \mu, f \rangle := \int_{\mathcal{G}} f(s) d\mu(s)$$

for all $\mu \in M(\mathcal{G})$ and $f \in C_0(\mathcal{G})$. This allows us to view $M(\mathcal{G})$ as a subalgebra of $LUC(\mathcal{G})^*$.

Throughout the paper the letter $M^+(\mathcal{G})$ means the set of all positive measures in $M(\mathcal{G})$; δ_s denotes the Dirac measure at $s \in \mathcal{G}$; and as usual, $M_a(\mathcal{G})$ denotes the closed ideal of $M(\mathcal{G})$ consisting of all absolutely continuous measures with respect to λ . Let also $L^1(\mathcal{G})$ denote the group algebra of \mathcal{G} as defined in [8]. Then, the Radon-Nikodym theorem can be interpreted as an identification of $M_a(\mathcal{G})$ with $\{\lambda_\varphi : \varphi \in L^1(\mathcal{G})\}$, where λ_φ is the measure in $M(\mathcal{G})$ defined on each Borel subset B of \mathcal{G} by

$$\lambda_\varphi(B) = \int_B \varphi d\lambda.$$

Recall that, the convolution of two functions $\varphi, \psi \in L^1(\mathcal{G})$ is the function defined by

$$\begin{aligned} \varphi * \psi(s) &= \int_{\mathcal{G}} \varphi(t) \psi(t^{-1}s) d\lambda(t) \\ &= \int_{\mathcal{G}} \varphi(st^{-1}) \psi(t) \Delta(t^{-1}) d\lambda(t). \end{aligned}$$

3 Definitions and some basic results

Recall that a locally compact Hausdorff space \mathcal{X} is said to be a (left) \mathcal{G} -space if there is a continuous action map

$$\mathcal{G} \times \mathcal{X} \rightarrow \mathcal{X}; (s, x) \mapsto s \cdot x,$$

satisfying $(st) \cdot x = s \cdot (t \cdot x)$ and $e \cdot x = x$, for all $s, t \in \mathcal{G}$ and $x \in \mathcal{X}$. In some published articles the terms (left) transformation group, a dynamical system or a flow are also used for the pair $(\mathcal{X}, \mathcal{G})$. Similarly one can define a right action and a right \mathcal{G} -space. Through this paper, when we refer to \mathcal{X} as a \mathcal{G} -space without any explicit reference to the left or right action, we mean \mathcal{X} is a left \mathcal{G} -space.

A \mathcal{G} -space \mathcal{X} is called transitive if for each $x, y \in \mathcal{X}$ there exists an element $s \in \mathcal{G}$ such that $y = s \cdot x$. Also, in the case where \mathcal{X} is a \mathcal{G} -space which is topologically isomorphic to \mathcal{G}/\mathcal{H} , for some closed subgroup \mathcal{H} of \mathcal{G} , we say \mathcal{X} is a homogeneous \mathcal{G} -space.

The action of \mathcal{G} on \mathcal{X} is said to be *free*, where $s \cdot x \neq x$ for every $x \in \mathcal{X}$ and $s \in \mathcal{G}$ with $s \neq e$. It is called *effective*, when the following closed normal subgroup of \mathcal{G} is trivial

$$\mathfrak{N}(\mathcal{X}, \mathcal{G}) = \{s \in \mathcal{G} : s \cdot x = x \text{ for all } x \in \mathcal{X}\}.$$

Similar to the previous section, we consider the Banach spaces $C_b(\mathcal{X})$ and $C_0(\mathcal{X})$. We also denote by $M(\mathcal{X})$ the Banach space of all complex Radon measures on \mathcal{X} with total variation norm. For given measures $\mu \in M(\mathcal{G})$ and $\sigma \in M(\mathcal{X})$ we may define their convolution $\mu \star \sigma$, as an element of $M(\mathcal{X})$, by

$$\langle \mu \star \sigma, F \rangle = \int_{\mathcal{G}} \int_{\mathcal{X}} F(s \cdot x) d\sigma(x) d\mu(s) \quad (F \in C_0(\mathcal{X})).$$

This allows us to consider $M(\mathcal{X})$ as a Banach left- $M(\mathcal{G})$ -module. Now let \mathcal{A} be a closed subalgebra of $M(\mathcal{G})$, then as usual, we say that $M(\mathcal{X})$ is a faithful Banach left \mathcal{A} -module, if the action of each $0 \neq \mu \in \mathcal{A}$ is non-trivial; that is, if $\mu \in \mathcal{A}$ is so that $\mu \star \sigma = 0$ for all $\sigma \in M(\mathcal{X})$, then $\mu = 0$. In the following result, we are concerned with relations between faithfulness of the action of $M(\mathcal{G})$ on $M(\mathcal{X})$ and the action of \mathcal{G} on \mathcal{X} . In a special case, it gives some conditions in relation to Veech's Theorem [18] in terms of the faithfulness of the action of $M(\mathcal{G})$ on $M(\mathcal{UG})$ which is a functional property, where \mathcal{UG} is the largest semigroup compactification of \mathcal{G} .

Let $(f_\alpha)_\alpha$ be a net in $C_b(\mathcal{G})$. We say $(f_\alpha)_\alpha$ converges to some $f \in C_b(\mathcal{G})$ strictly if $\|f_\alpha g - fg\|_\infty \rightarrow 0$, for all g in $S_0^+(\mathcal{G})$ the set of all non-negative upper semicontinuous real-valued functions on \mathcal{G} which vanish at infinity, see [1] for more details.

Theorem 3.1 *Let \mathcal{G} be a locally compact group and \mathcal{X} be a \mathcal{G} -space. Then the following statements hold.*

- (a) *The Banach space $M(\mathcal{X})$ is a faithful Banach left $M^+(\mathcal{G})$ -module if and only if \mathcal{G} acts effectively on \mathcal{X} .*
- (b) *If for some $x_0 \in \mathcal{G}$ the stabilizer subgroup $\mathfrak{N}(x_0, \mathcal{G}) = \{s \in \mathcal{G} : s \cdot x_0 = x_0\}$ of \mathcal{G} is trivial, then $M(\mathcal{X})$ is a faithful Banach left $M(\mathcal{G})$ -module. In particular, if \mathcal{G} acts freely on \mathcal{X} , then $M(\mathcal{G})$ acts faithfully on $M(\mathcal{X})$.*
- (c) *If $M_a(\mathcal{G})$ acts faithfully on $M(\mathcal{X})$, then $M(\mathcal{X})$ is a faithful Banach left $M(\mathcal{G})$ -module.*

Proof. (a) First, suppose that $M(\mathcal{X})$ is a faithful Banach left $M^+(\mathcal{G})$ -module and $s \in \mathfrak{N}(\mathcal{X}, \mathcal{G})$. Then for all $\sigma \in M(\mathcal{X})$ we have

$$\langle \delta_s \star \sigma, F \rangle = \int_{\mathcal{X}} F(s \cdot x) d\sigma(x) = \int_{\mathcal{X}} F(x) d\sigma(x) = \langle \sigma, F \rangle, \quad (F \in C_0(\mathcal{X}))$$

Accordingly, $\delta_s \star \sigma = \sigma$ for all $\sigma \in M(\mathcal{X})$. This together with the fact that $M^+(\mathcal{G})$ acts faithfully on $M(\mathcal{X})$ implies that $\delta_s = \delta_e$ and therefore $s = e$.

Conversely, assume that $\mathfrak{N}(\mathcal{X}, \mathcal{G}) = \{e\}$ and μ is a positive measure in $M(\mathcal{G})$ such that $\mu \star \sigma = 0$ for all $\sigma \in M(\mathcal{X})$. Observe that $\nu = \mu + \delta_e$ is a positive measure in $M(\mathcal{G})$ for which $\nu \star \sigma = \sigma$ for all $\sigma \in M(\mathcal{X})$. Hence, the proof will be completed by showing that $\text{supp}(\nu) = \{e\}$ or equivalently $\nu = \delta_e$. To this end, suppose on the contrary that s_0 is an element in $\text{supp}(\nu) \setminus \{e\}$. This, together with the fact that $\mathfrak{N}(\mathcal{X}, \mathcal{G}) = \{e\}$ implies that there exists some $x_0 \in \mathcal{X}$ such that $s_0 \cdot x_0 \neq x_0$. Taking a positive function F in $C_c(\mathcal{X})$ with $F(x_0) = 0$ and $F(s_0 \cdot x_0) = 1$, we have

$$(\nu \star \delta_{x_0})(F) = \delta_{x_0}(F) = F(x_0) = 0.$$

On the other hand, there is an open neighborhood U of s_0 such that $F(s \cdot x_0) > 1/2$ for all $s \in U$. As $s_0 \in \text{supp}(\nu)$, we have $\nu(U) > 0$ and hence, we can write

$$0 = (\nu \star \delta_{x_0})(F) = \int_{\mathcal{G}} F(s \cdot x_0) d\nu(s) \geq \nu(U)/2,$$

which is a contradiction. Thus, we get $\text{supp}(\nu) = \{e\}$. It follows that $\nu = \delta_e$ and therefore $\mu = 0$.

(b) Assume that $\mathfrak{N}(x_0, \mathcal{G}) = \langle e \rangle$, for some $x_0 \in \mathcal{X}$. Obviously, for all $F \in C_b(\mathcal{X})$, the function $r_{x_0}F : \mathcal{G} \rightarrow \mathbb{C}$ defined by $r_{x_0}F(s) = F(s \cdot x_0)$ belongs to $C_b(\mathcal{G})$. Moreover,

$$\mathcal{A} := \left\{ r_{x_0}F : F \in C_b(\mathcal{X}) \right\},$$

is a self-adjoint subalgebra of $C_b(\mathcal{G})$ such that for each $s \in \mathcal{G}$, there exists some $F \in C_b(\mathcal{X})$ for which $r_{x_0}F(s) = F(s \cdot x_0) \neq 0$. Furthermore, this subalgebra separates the points of \mathcal{G} , since for all s_1 and s_2 in \mathcal{G} with $s_1 \neq s_2$ we have $s_1 \cdot x_0 \neq s_2 \cdot x_0$ and therefore there exists $F \in C_b(\mathcal{X})$ such that

$$r_{x_0}F(s_1) = F(s_1 \cdot x_0) \neq F(s_2 \cdot x_0) = r_{x_0}F(s_2).$$

From this, by a generalized version of the Stone-Weierstrass theorem [1, Theorem 3.2], we can conclude that \mathcal{A} is strictly dense in $C_b(\mathcal{G})$.

Now, let μ be a measure in $M(\mathcal{G})$ such that $\mu \star \sigma = 0$ for all $\sigma \in M(\mathcal{X})$ and $g \in C_c(\mathcal{G})$ be given. Then, by taking $\sigma = \delta_{x_0}$, we have

$$\int_{\mathcal{G}} F(s \cdot x_0) d\mu(s) = \langle \mu \star \delta_{x_0}, F \rangle = 0, \quad (2)$$

for all $F \in C_b(\mathcal{X})$. Take a neighborhood V of $\text{supp}(g)$ with compact closure and some $F' \in C_c^+(\mathcal{X})$ such that

$$F'|_{\text{supp}(g) \cdot x_0} = 1 \quad \text{and} \quad F'|_{\mathcal{X} \setminus \overline{V} \cdot x_0} = 0.$$

Therefore, $r_{x_0}F'$ is a non-negative continuous function for which

$$(r_{x_0}F')|_{\text{supp}(g)} = 1 \quad \text{and} \quad (r_{x_0}F')|_{\mathcal{X} \setminus \overline{V}} = 0;$$

This is because of, for all $s \in \mathcal{G}$, the condition $s \notin \overline{V}$ implies that $s \cdot x_0 \notin \overline{V} \cdot x_0$. So, $r_{x_0}F' \in C_c^+(\mathcal{G})$ and hence by taking a sequence $(F_n)_{n \in \mathbb{N}} \subseteq C_b(\mathcal{X})$ for which $(r_{x_0}F_n)_{n \in \mathbb{N}}$ tends strictly to g , we can write $\|(r_{x_0}F_n)(r_{x_0}F') - g(r_{x_0}F')\|_{\infty} \rightarrow 0$. Consequently, we have

$$\begin{aligned} \int_{\mathcal{G}} g(s) d\mu(s) &= \int_{\mathcal{G}} g(s)(r_{x_0}F')(s) d\mu(s) \\ &= \langle \mu, (r_{x_0}F')g \rangle \\ &= \lim_{n \rightarrow +\infty} \langle \mu, (r_{x_0}F_n)(r_{x_0}F') \rangle \\ &= \lim_{n \rightarrow +\infty} \int_{\mathcal{G}} F_n(s \cdot x_0) F'(s \cdot x_0) d\mu(s) \\ &= \lim_{n \rightarrow +\infty} \int_{\mathcal{G}} (F_n F')(s \cdot x_0) d\mu(s). \end{aligned}$$

As $F_n F' \in C_b(\mathcal{X})$, by using (2), we deduce $\int_{\mathcal{G}} g(s) d\mu(s) = 0$. Now, since g is arbitrary in $C_c(\mathcal{G})$, we can deduce that $\mu = 0$ and this completes the proof of this part.

(b) Let μ be a measure in $M(\mathcal{G})$ such that $\mu \star \sigma = 0$ for all $\sigma \in M(\mathcal{X})$. Recall from [9, Lemma 2.1. and Lemma 2.2.] that the algebra $L^1(\mathcal{G})$ possesses a bounded approximate identity (φ_{α}) such that each φ_{α} is a continuous function with compact support and $\lambda_{\varphi_{\alpha}} \rightarrow \delta_e$ in the weak* topology of $M(\mathcal{G})$. As $M_a(\mathcal{G})$ is an ideal of $M(\mathcal{G})$, we have $\lambda_{\varphi_{\alpha}} * \mu \in M_a(\mathcal{G})$ and also

$$(\lambda_{\varphi_{\alpha}} * \mu) \star \sigma = \lambda_{\varphi_{\alpha}} \star (\mu \star \sigma) = 0,$$

for all $\sigma \in M(\mathcal{X})$ and all α . Accordingly, from the assumption, we get $\lambda_{\varphi_\alpha} * \mu = 0$ for all α . It follows that $\mu = 0$. \blacksquare

When \mathcal{X} is a \mathcal{G} -space, by the use of the action of \mathcal{G} on \mathcal{X} , we introduce a certain subspace of $C_b(\mathcal{X})$ which is the main object of study of this work. In details, following Lau and Chu [4] for a given s in \mathcal{G} , we define the left translation of a function $F \in C_b(\mathcal{X})$, by an element $s \in \mathcal{G}$, by $l_s F(x) = F(s \cdot x)$ for all $x \in \mathcal{X}$ and consider

$$LUC(\mathcal{X}, \mathcal{G}) = \left\{ F \in C_b(\mathcal{X}) : s \in \mathcal{G} \mapsto l_s F \in (C_b(\mathcal{X}), \|\cdot\|_\infty) \text{ is continuous} \right\}.$$

Then $LUC(\mathcal{X}, \mathcal{G})$ is a closed subspace of $C_b(\mathcal{X})$ which is invariant under left translation. We always let \mathcal{G} act on itself by left translation and therefore $LUC(\mathcal{G}) = LUC(\mathcal{G}, \mathcal{G})$. It is well-known that $C_0(\mathcal{G}) \subseteq LUC(\mathcal{G})$ (see [8, Proposition 2.6]), by the same argument one can prove the following result, which guarantees that $LUC(\mathcal{X}, \mathcal{G})$ contains enough elements to separate the points of \mathcal{X} .

Lemma 3.2 *Let \mathcal{G} be a locally compact group and \mathcal{X} be a \mathcal{G} -space. Then the Banach space $C_0(\mathcal{X})$ is contained in $LUC(\mathcal{X}, \mathcal{G})$.*

Moreover, if $LUC(\mathcal{X}, \mathcal{G})^*$ denotes the first dual space of the Banach space $LUC(\mathcal{X}, \mathcal{G})$, M is an arbitrary element of $LUC(\mathcal{X}, \mathcal{G})^*$ and $F \in LUC(\mathcal{X}, \mathcal{G})$, we define the function $MF : \mathcal{G} \rightarrow \mathbb{C}$ by

$$MF(s) = \langle M, l_s F \rangle,$$

for all $s \in \mathcal{G}$, which belongs to $LUC(\mathcal{G})$; This is because of,

$${}_s(MF)(t) = (MF)(st) = \langle M, l_{st} F \rangle = \langle M, l_t(l_s F) \rangle = (M(l_s F))(t), \quad (3)$$

for all $s, t \in \mathcal{G}$. Also we have the following lemma which plays a key role in this paper.

Lemma 3.3 *Let \mathcal{G} be a locally compact group and \mathcal{X} be a \mathcal{G} -space. Then the mapping*

$$\begin{cases} LUC(\mathcal{X}, \mathcal{G})^* \times LUC(\mathcal{X}, \mathcal{G}) \rightarrow LUC(\mathcal{G}) \\ (M, F) \mapsto MF \end{cases}$$

is a bounded bilinear map with $\|MF\|_\infty \leq \|M\| \|F\|_\infty$.

Lemma 3.3 paves the way for defining a bounded bilinear map as follows

$$\begin{cases} LUC(\mathcal{G})^* \times LUC(\mathcal{X}, \mathcal{G})^* \rightarrow LUC(\mathcal{X}, \mathcal{G})^* \\ (m, M) \mapsto m \cdot M \end{cases}$$

with $\|m \cdot M\| \leq \|m\| \|M\|$, where

$$\langle m \cdot M, F \rangle = \langle m, MF \rangle, \quad (4)$$

for all $F \in LUC(\mathcal{X}, \mathcal{G})$.

Proposition 3.4 *Let \mathcal{G} be a locally compact group and \mathcal{X} be a \mathcal{G} -space. Under the mapping $(m, M) \mapsto m \cdot M$, defined by (4), $LUC(\mathcal{X}, \mathcal{G})^*$ becomes a Banach left $LUC(\mathcal{G})^*$ -module with $\|m \cdot M\| \leq \|m\| \|M\|$ and $\delta_e \cdot M = M$.*

Proof. It is enough to prove that $(m \odot n) \cdot M = m \cdot (n \cdot M)$, where $m, n \in LUC(\mathcal{G})^*$ and $M \in LUC(\mathcal{X}, \mathcal{G})^*$. To this end, suppose that $F \in LUC(\mathcal{X}, \mathcal{G})$. Then, by (1) and (3), we can see that for all $s \in \mathcal{G}$

$$(n \diamond (MF))(s) = \langle n, {}_s(MF) \rangle = \langle n, M(l_s F) \rangle = \langle n \cdot M, l_s F \rangle = ((n \cdot M)F)(s),$$

which shows that $n \diamond (MF) = (n \cdot M)F$. Therefore,

$$\begin{aligned} \langle (m \odot n) \cdot M, F \rangle &= \langle m \odot n, MF \rangle \\ &= \langle m, n \diamond (MF) \rangle \\ &= \langle m, (n \cdot M)F \rangle \\ &= \langle m \cdot (n \cdot M), F \rangle, \end{aligned}$$

and this completes the proof. ■

If now, for a given $\sigma \in M(\mathcal{X})$, we define a linear functional on $LUC(\mathcal{X}, \mathcal{G})$, denoted again by σ , which assigns to each $F \in LUC(\mathcal{X}, \mathcal{G})$ the value $\int_{\mathcal{X}} F(x) d\sigma(x)$. Then, $M(\mathcal{X})$ may be regarded as a subspace of $LUC(\mathcal{X}, \mathcal{G})^*$. Moreover, it is not hard to check that the inclusion $LUC(\mathcal{G})^* \cdot M(\mathcal{X}) \subseteq M(\mathcal{X})$ can fail even if $\mathcal{X} = \mathcal{G}$, where

$$LUC(\mathcal{G})^* \cdot M(\mathcal{X}) = \{m \cdot \sigma : m \in LUC(\mathcal{G})^*, \sigma \in M(\mathcal{X})\};$$

In other word, the Banach space $M(\mathcal{X})$ is not in general an $LUC(\mathcal{G})^*$ -submodule of $LUC(\mathcal{X}, \mathcal{G})^*$. On the other hand, if \mathcal{X} is compact, then $M(\mathcal{X})$ is an $LUC(\mathcal{G})^*$ -submodule of $LUC(\mathcal{X}, \mathcal{G})^*$; This is because of, in this case $M(\mathcal{X}) = LUC(\mathcal{X}, \mathcal{G})^*$. We do not know if the converse of this fact is valid in general; here, we prove the converse under an extra assumption. The notation $\mathcal{CLS}(\mathcal{X}, \mathcal{G})$ in this proposition and in the sequel denotes the norm closure of the linear span of the set

$$LUC(\mathcal{X}, \mathcal{G})^* LUC(\mathcal{X}, \mathcal{G}) := \{MF : M \in LUC(\mathcal{X}, \mathcal{G})^*, F \in LUC(\mathcal{X}, \mathcal{G})\},$$

with respect to the norm topology of $LUC(\mathcal{G})$. Also, we say that the action of $LUC(\mathcal{G})^*$ on $LUC(\mathcal{X}, \mathcal{G})^*$ is faithful if $m \in LUC(\mathcal{G})^*$ is so that $m \cdot M = 0$ for all $M \in LUC(\mathcal{X}, \mathcal{G})^*$, then $m = 0$.

Proposition 3.5 *Let \mathcal{G} be a locally compact group and \mathcal{X} be a \mathcal{G} -space. Then the following statements hold.*

- (a) *The action of $LUC(\mathcal{G})^*$ on $LUC(\mathcal{X}, \mathcal{G})^*$ is faithful if and only if $\mathcal{CLS}(\mathcal{X}, \mathcal{G}) = LUC(\mathcal{G})$.*
- (b) *If the action of $LUC(\mathcal{G})^*$ on $LUC(\mathcal{X}, \mathcal{G})^*$ is faithful, then \mathcal{G} acts effectively on \mathcal{X} .*
- (c) *If \mathcal{X} is compact, then $M(\mathcal{X})$ is an $LUC(\mathcal{G})^*$ -submodule of $LUC(\mathcal{X}, \mathcal{G})^*$. The converse is also true if \mathcal{X} is a transitive \mathcal{G} -space.*

Proof. **(a)** The proof of this assertion uses only the Hahn-Banach theorem and so the details are omitted.

(b) If the action of $LUC(\mathcal{G})^*$ on $LUC(\mathcal{X}, \mathcal{G})^*$ is faithful and $s \in \mathfrak{N}(\mathcal{X}, \mathcal{G})$, then for all $M \in LUC(\mathcal{X}, \mathcal{G})^*$ and $F \in LUC(\mathcal{X}, \mathcal{G})$ we have

$$\langle \delta_s \cdot M, F \rangle = \langle \delta_s, MF \rangle = MF(s) = \langle M, l_s F \rangle = \langle M, F \rangle.$$

So, $\delta_s \cdot M = M$ for all $M \in LUC(\mathcal{X}, \mathcal{G})^*$ and hence, $\delta_s = \delta_e$; that is, $s = e$.

(c) We only need to prove the converse of this assertion, which is the essential part of it. To this end, suppose that $M(\mathcal{X})$ is a submodule of $LUC(\mathcal{X}, \mathcal{G})^*$. Then for all $m \in LUC(\mathcal{G})^*$ and $\sigma \in M(\mathcal{X})$, we have $m \cdot \sigma \in M(\mathcal{X})$. This implies that the norm of the linear functional $m \cdot \sigma$ on $LUC(\mathcal{X}, \mathcal{G})$ is obtained by the following equality

$$\|m \cdot \sigma\| = \sup \left\{ |\langle m \cdot \sigma, F_0 \rangle| : F_0 \in C_0(\mathcal{X}), \|F_0\|_\infty = 1 \right\}.$$

Therefore,

$$|\langle m \cdot \sigma, F \rangle| \leq \sup \left\{ |\langle m \cdot \sigma, F_0 \rangle| : F_0 \in C_0(\mathcal{X}), \|F_0\|_\infty = 1 \right\},$$

for all $F \in LUC(\mathcal{X}, \mathcal{G})$ with $\|F\|_\infty = 1$. Hence, for all $m \in LUC(\mathcal{G})^*$, $\sigma \in M(\mathcal{X})$, and $F \in LUC(\mathcal{X}, \mathcal{G})$ with $\|F\|_\infty = 1$ we have

$$|\langle m, \sigma F \rangle| \leq \sup \left\{ |\langle m, \sigma F_0 \rangle| : F_0 \in C_0(\mathcal{X}), \|F_0\|_\infty = 1 \right\}. \quad (5)$$

From this, we deduce that, for all $\sigma \in M(\mathcal{X})$ and $F \in LUC(\mathcal{X}, \mathcal{G})$, σF is in the norm closure of the linear span of the set

$$\sigma C_0(\mathcal{X}) := \left\{ \sigma F_0 : F_0 \in C_0(\mathcal{X}) \right\},$$

with respect to the norm topology of $LUC(\mathcal{G})$; since, otherwise, we can find a $m \in LUC(\mathcal{G})^*$ so that $\langle m, \sigma F \rangle = 1$ and $\langle m, \sigma F_0 \rangle = 0$ for all $F_0 \in C_0(\mathcal{X})$, which is a contradiction to (5). In particular, if x and F are arbitrary elements of \mathcal{X} and $LUC(\mathcal{X}, \mathcal{G})$, respectively, then, there exists a sequence $(F_n)_{n \in \mathbb{N}} \subseteq C_0(\mathcal{X})$ so that $\|\delta_x F_n - \delta_x F\|_\infty \rightarrow 0$. Now from the transitivity of the action of \mathcal{G} on \mathcal{X} , we get

$$\begin{aligned} \|F_n - F\|_\infty &\leq \sup_{s \in \mathcal{G}} |F_n(s \cdot x) - F(s \cdot x)| \\ &\leq \|\delta_x F_n - \delta_x F\|_\infty \\ &= \sup_{s \in \mathcal{G}} |(\delta_x F_n)(s) - (\delta_x F)(s)|. \end{aligned}$$

Thus, $\|F_n - F\|_\infty \rightarrow 0$, which implies that F is in $C_0(\mathcal{X})$. Therefore, $LUC(\mathcal{X}, \mathcal{G}) \subseteq C_0(\mathcal{X})$. As $LUC(\mathcal{X}, \mathcal{G})$ contains the constant functions on \mathcal{X} , we deduce that \mathcal{X} is compact. \blacksquare

4 The weak* continuity of the left $LUC(\mathcal{G})^*$ -module action

As we know, Lau [14] has shown that a left translation $n \mapsto m \odot n$ is weak* continuous on $LUC(\mathcal{G})^*$ for a fixed m in $LUC(\mathcal{G})^*$ if and only if m is in $M(\mathcal{G})$; that is, $\mathfrak{Z}(\mathcal{G})$, the topological centre of $LUC(\mathcal{G})^*$, is $M(\mathcal{G})$. It follows that, if \mathcal{X} is a \mathcal{G} -space, then for an arbitrary m in

$LUC(\mathcal{G})^*$ the weak* continuity of the map $M \mapsto m \cdot M$ on $LUC(\mathcal{X}, \mathcal{G})^*$ can fail even if $\mathcal{X} = \mathcal{G}$. A problem which is of interest is that for which element $m \in LUC(\mathcal{G})$ the map $M \mapsto m \cdot M$ on $LUC(\mathcal{X}, \mathcal{G})^*$ is weak* to weak* continuous? Therefore, it seems valuable to define

$$\mathfrak{Z}(\mathcal{X}, \mathcal{G}) = \left\{ m \in LUC(\mathcal{G})^* : M \mapsto m \cdot M \text{ is weak}^* \text{ to weak}^* \text{ continuous on } LUC(\mathcal{X}, \mathcal{G})^* \right\},$$

the topological centre of the module action induced by $LUC(\mathcal{G})^*$ on $LUC(\mathcal{X}, \mathcal{G})^*$. In the special case that we let \mathcal{G} act on itself by left multiplication, the set $\mathfrak{Z}(\mathcal{G}, \mathcal{G})$ coincides with $\mathfrak{Z}(\mathcal{G})$, which defined in Section 1.

This section studies the subspace $\mathfrak{Z}(\mathcal{X}, \mathcal{G})$ of $LUC(\mathcal{G})^*$ in the case where \mathcal{X} is a \mathcal{G} -space and, in particular, the question when the subspace $\mathfrak{Z}(\mathcal{X}, \mathcal{G})$ is $M(\mathcal{G})$ or $LUC(\mathcal{G})^*$. Before proceeding further in this section, we should note that if \mathcal{X} is a \mathcal{G} -space, then $LUC(\mathcal{X}, \mathcal{G})^*$ is a left Banach \mathcal{G} -module. In fact, it suffices to define the left action of \mathcal{G} on $LUC(\mathcal{X}, \mathcal{G})^*$ by $(s, M) \mapsto \delta_s \cdot M$.

Lemma 4.1 *Let \mathcal{G} be a locally compact group and \mathcal{X} be a \mathcal{G} -space. Then the action of \mathcal{G} on the unit ball of $LUC(\mathcal{X}, \mathcal{G})^*$ with the weak* topology is jointly continuous.*

Proof. For the proof, it suffices to note that if (s_α) is a net converging to s in \mathcal{G} , then the net (δ_{s_α}) tends to δ_s with respect to the weak* topology of $LUC(\mathcal{G})^*$. \blacksquare

Let \mathcal{X} be a \mathcal{G} -space. Given $F \in LUC(\mathcal{X}, \mathcal{G})$ and $x \in \mathcal{X}$, we define $r_x F$ on \mathcal{G} by

$$(r_x F)(s) = F(s \cdot x), \quad (s \in \mathcal{G}),$$

then a routine computation shows that

$$\|s_\alpha(r_x F) - s(r_x F)\|_\infty \leq \|l_{s_\alpha} F - l_s F\|_\infty,$$

where (s_α) is a net in \mathcal{G} which tends to s , and therefore $r_x F$ is a function in $LUC(\mathcal{G})$. Hence, if m is an arbitrary element of $LUC(\mathcal{G})^*$, then we can define a complex-valued function Fm on \mathcal{X} by

$$Fm(x) = \langle m, r_x F \rangle \quad (x \in \mathcal{X}).$$

Observe that Fm is a bounded function on \mathcal{X} with $\|Fm\|_\infty \leq \|m\| \|F\|_\infty$. The following lemma shows that the continuity of the function Fm can fail even if $\mathcal{X} = \mathcal{G}$.

Lemma 4.2 *Let \mathcal{G} be a locally compact group and \mathcal{X} be a \mathcal{G} -space. Then, for all $m \in LUC(\mathcal{G})^*$ and $F \in LUC(\mathcal{X}, \mathcal{G})$, the following assertions hold.*

- (a) For all $x \in \mathcal{X}$, $Fm(x) = \langle m \cdot \delta_x, F \rangle$.
- (b) If $m \in \mathfrak{Z}(\mathcal{X}, \mathcal{G})$, then Fm is in $C_b(\mathcal{X})$.
- (c) If $m \in \mathfrak{Z}(\mathcal{X}, \mathcal{G})$, then $\langle \delta_x, l_s(Fm) \rangle = \langle (m \odot \delta_s) \cdot \delta_x, F \rangle$ for all $x \in \mathcal{X}$ and $s \in \mathcal{G}$.

Proof. Let $m \in LUC(\mathcal{G})^*$, $F \in LUC(\mathcal{X}, \mathcal{G})$, $x \in \mathcal{X}$ and $s \in \mathcal{G}$ be given.

(a) First, observe that

$$(\delta_x F)(s) = \langle \delta_x, l_s F \rangle = l_s F(x) = F(s \cdot x) = r_x F(s), \quad (6)$$

for all $s \in \mathcal{G}$. Hence, we have

$$\langle m \cdot \delta_x, F \rangle = \langle m, \delta_x F \rangle = \langle m, r_x F \rangle = Fm(x).$$

(b) If $m \in \mathfrak{Z}(\mathcal{X}, \mathcal{G})$ and $(x_\alpha) \subseteq \mathcal{X}$ converges to $x \in \mathcal{X}$, then the net (δ_{x_α}) tends to δ_x in the weak* topology of $LUC(\mathcal{X}, \mathcal{G})^*$. Therefore, since $m \in \mathfrak{Z}(\mathcal{X}, \mathcal{G})$, we have

$$(Fm)(x_\alpha) = \langle m \cdot \delta_{x_\alpha}, F \rangle \longrightarrow \langle m \cdot \delta_x, F \rangle = (Fm)(x).$$

This together with the fact that $\|Fm\|_\infty \leq \|m\| \|F\|_\infty$, implies that $Fm \in C_b(\mathcal{X})$.

(c) In this case, $r_x F$ is a function in $LUC(\mathcal{G})$, $Fm \in C_b(\mathcal{X})$ and for each $t \in \mathcal{G}$ we have

$$\delta_s \diamond (r_x F)(t) = \langle \delta_s, {}_t(r_x F) \rangle = {}_t(r_x F)(s) = F(t \cdot (s \cdot x)) = (r_{s \cdot x} F)(t),$$

From this, by (6), we deduce that

$$\langle \delta_x, l_s(Fm) \rangle = Fm(s \cdot x) = \langle m, r_{s \cdot x} F \rangle = \langle m \odot \delta_s, r_x F \rangle = \langle (m \odot \delta_s) \cdot \delta_x, F \rangle,$$

as desired. ■

Now, suppose that m is an element of $LUC(\mathcal{G})^*$ such that, for each $F \in LUC(\mathcal{X}, \mathcal{G})$, the function Fm is in $LUC(\mathcal{X}, \mathcal{G})$. Then every M in $LUC(\mathcal{X}, \mathcal{G})^*$ gives a linear functional $M \bullet m$ on $LUC(\mathcal{X}, \mathcal{G})$ as follows

$$\langle M \bullet m, F \rangle = \langle M, Fm \rangle.$$

In addition, if $M \bullet m = m \cdot M$, for all $M \in LUC(\mathcal{X}, \mathcal{G})^*$, then $m \in \mathfrak{Z}(\mathcal{X}, \mathcal{G})$; Indeed, if $(M_\alpha) \subseteq LUC(\mathcal{X}, \mathcal{G})^*$ tends to an element M with respect to the weak* topology, then we have

$$\lim_\alpha \langle m \cdot M_\alpha, F \rangle = \lim_\alpha \langle M_\alpha \bullet m, F \rangle = \langle M, Fm \rangle = \langle M \bullet m, F \rangle = \langle m \cdot M, F \rangle,$$

for all $F \in LUC(\mathcal{X}, \mathcal{G})$. The following theorem consider the converse of this fact whose proof is inspired by [14, Lemma 2.2].

Theorem 4.3 *Let \mathcal{G} be a locally compact group and \mathcal{X} be a \mathcal{G} -space. Then $\mathfrak{Z}(\mathcal{X}, \mathcal{G})$ is precisely the set of all $m \in LUC(\mathcal{G})^*$ for which the following conditions are satisfied*

- (a) $Fm \in LUC(\mathcal{X}, \mathcal{G})$ for all $F \in LUC(\mathcal{X}, \mathcal{G})$,
- (b) $M \bullet m = m \cdot M$ for all $M \in LUC(\mathcal{X}, \mathcal{G})^*$.

Proof. Suppose that $m \in \mathfrak{Z}(\mathcal{X}, \mathcal{G})$ and F is an arbitrary element of $LUC(\mathcal{X}, \mathcal{G})$. Then, the map $M \mapsto m \cdot M$ is weak* continuous on $LUC(\mathcal{X}, \mathcal{G})^*$ and therefore it is weak* continuous on bounded sets. Also, part (b) of Lemma 4.2 implies that Fm is a function in $C_b(\mathcal{X})$. Moreover, in light of Lemma 4.1, part (c) of Lemma 4.2, [3, Lemma 2.5] and the weak* continuity of the map $M \mapsto m \cdot M$ on bounded set, we have

$$\langle \Lambda, l_s(Fm) \rangle = \langle (m \odot \delta_s) \cdot \Lambda, F \rangle \tag{7}$$

for all $\Lambda \in C_b(\mathcal{X})^*$ and $s \in \mathcal{G}$. Now, suppose that (s_α) is a net in \mathcal{G} converging to s and for each α the functional $\Lambda_\alpha \in C_b(\mathcal{X})^*$ is chosen so that $\|\Lambda_\alpha\| = 1$ and

$$\|l_{s_\alpha}(Fm) - l_s(Fm)\|_\infty = \left| \langle \Lambda_\alpha, l_{s_\alpha}(Fm) - l_s(Fm) \rangle \right|.$$

Let also, Λ be a weak*-cluster point of (Λ_α) . By passing to a subnet, if necessary, we may assume that $\Lambda_\alpha \rightarrow \Lambda$ in the weak* topology of $LUC(\mathcal{X}, \mathcal{G})^*$. Hence

$$\begin{aligned} \|l_{s_\alpha}(Fm) - l_s(Fm)\|_\infty &\leq \left| \langle (m \odot \delta_{s_\alpha}) \cdot \Lambda_\alpha - (m \odot \delta_s) \cdot \Lambda, F \rangle \right| \\ &\quad + \left| \langle (m \odot \delta_s) \cdot \Lambda - (m \odot \delta_s) \cdot \Lambda_\alpha, F \rangle \right|. \end{aligned}$$

Now, the weak* continuity of the map $M \mapsto m \cdot M$ on norm bounded subsets of $LUC(\mathcal{X}, \mathcal{G})^*$ and Lemma 4.1 imply that Fm is in $LUC(\mathcal{X}, \mathcal{G})$.

Finally, to prove the equality $M \bullet m = m \cdot M$ for all $M \in LUC(\mathcal{X}, \mathcal{G})^*$. First note that $\delta_x \bullet m = m \cdot \delta_x$, for all $x \in \mathcal{X}$. Hence, the equality $M \bullet m = m \cdot M$ holds if M is a convex combination of the Dirac measures. We now invoke [3, Lemma 2.5] to conclude that $M \bullet m = m \cdot M$ holds for all $M \in LUC(\mathcal{X}, \mathcal{G})^*$. \blacksquare

It is clear that if $m \in \mathfrak{Z}(\mathcal{X}, \mathcal{G})$, then the map $M \mapsto m \cdot M$ is weak* continuous on all bounded parts of $LUC(\mathcal{X}, \mathcal{G})^*$. As we have seen in the proof of Theorem 4.3, that the map $M \mapsto m \cdot M$ is weak* continuous on all bounded parts of $LUC(\mathcal{X}, \mathcal{G})^*$ plays a key role in its proof. Hence, with an argument similar to the proof of Theorem 4.3 one can prove the following result. The details are omitted.

Corollary 4.4 *Let \mathcal{G} be a locally compact group and \mathcal{X} be a \mathcal{G} -space. Then $m \in \mathfrak{Z}(\mathcal{X}, \mathcal{G})$ if and only if the map $M \mapsto m \cdot M$ is weak* continuous on all bounded parts of $LUC(\mathcal{X}, \mathcal{G})^*$.*

The following two propositions paves the way for obtaining the wide class of \mathcal{G} -space \mathcal{X} for which

$$M(\mathcal{G}) \subsetneq \mathfrak{Z}(\mathcal{X}, \mathcal{G}) = LUC(\mathcal{G})^*.$$

First let us recall that, if \mathcal{H} is a subgroup of \mathcal{G} , then the *index* of \mathcal{H} in \mathcal{G} , denoted by $[\mathcal{G} : \mathcal{H}]$, is the number of left cosets of \mathcal{H} in \mathcal{G} .

Proposition 4.5 *Let \mathcal{G} be a locally compact group and \mathcal{X} be a \mathcal{G} -space. If the index of $\mathfrak{N}(\mathcal{X}, \mathcal{G})$ in \mathcal{G} is finite, then $\mathfrak{Z}(\mathcal{X}, \mathcal{G}) = LUC(\mathcal{G})^*$.*

Proof. Let $\mathfrak{N} := \mathfrak{N}(\mathcal{X}, \mathcal{G})$ and m be an arbitrary element of $LUC(\mathcal{G})^*$. To prove that $m \in \mathfrak{Z}(\mathcal{X}, \mathcal{G})$, by Corollary 4.4 above, it will be enough to prove that the map $M \mapsto m \cdot M$ is weak* continuous on bounded parts of $LUC(\mathcal{X}, \mathcal{G})^*$. To this end, suppose that (M_α) is a bounded net in $LUC(\mathcal{X}, \mathcal{G})^*$ which tends to $M \in LUC(\mathcal{X}, \mathcal{G})^*$ with respect to the weak* topology. Then, since the index of \mathfrak{N} in \mathcal{G} is finite, we can obtain a partition $\{s_1\mathfrak{N}, s_2\mathfrak{N}, \dots, s_k\mathfrak{N}\}$ for \mathcal{G} and therefore for each $s \in \mathcal{G}$, there exists $t \in \mathfrak{N}$ and a unique $1 \leq i \leq k$ for which $s = s_i t$. It follows that for all $\alpha, s \in \mathcal{G}$ and $F \in LUC(\mathcal{X}, \mathcal{G})$

$$M_\alpha F(s) = \sum_{i=1}^k M_\alpha F(s_i) \chi_{s_i \mathfrak{N}}(s), \quad \text{and} \quad MF(s) = \sum_{i=1}^k MF(s_i) \chi_{s_i \mathfrak{N}}(s),$$

where $\chi_{s_i\mathfrak{N}}$ denotes the characteristic function of $s_i\mathfrak{N}$ on \mathcal{G} . On the other hand, since, for each $F \in LUC(\mathcal{X}, \mathcal{G})$, $M_\alpha F \rightarrow MF$ pointwisely, we may choose α_0 such that $|M_\alpha F(s_i) - MF(s_i)| < \varepsilon$ whenever $i = 1, \dots, k$ and $\alpha \succeq \alpha_0$. Therefore, for each $\alpha \succeq \alpha_0$, we have

$$\|M_\alpha F - MF\|_\infty = \left\| \sum_{i=1}^k M_\alpha F(s_i) \chi_{s_i\mathfrak{N}} - \sum_{i=1}^k MF(s_i) \chi_{s_i\mathfrak{N}} \right\|_\infty < \varepsilon.$$

This implies that

$$|\langle m \cdot M_\alpha, F \rangle - \langle m \cdot M, F \rangle| \leq \|m\| \|M_\alpha F - MF\|_\infty \rightarrow 0.$$

Therefore, $m \in \mathfrak{Z}(\mathcal{X}, \mathcal{G})$. ■

The following proposition, gives some of the main properties of the set $\mathfrak{Z}(\mathcal{X}, \mathcal{G})$.

Proposition 4.6 *Let \mathcal{G} be a locally compact group and \mathcal{X} be a \mathcal{G} -space. Then the following assertions hold.*

- (a) $\mathfrak{Z}(\mathcal{X}, \mathcal{G})$ is a subalgebra of $LUC(\mathcal{G})^*$.
- (b) $\mathfrak{Z}(\mathcal{X}, \mathcal{G})$ is closed with respect to the norm topology of $LUC(\mathcal{G})^*$.
- (c) $M(\mathcal{G})$ is contained in $\mathfrak{Z}(\mathcal{X}, \mathcal{G})$.

Proof. In order to prove (a), assume that m and n are in $\mathfrak{Z}(\mathcal{X}, \mathcal{G})$. Suppose also that F is an arbitrary element of $LUC(\mathcal{X}, \mathcal{G})$. Then, by Theorem 4.3, the functions $F' := Fm$ and $F'n$ are in $LUC(\mathcal{X}, \mathcal{G})$. On the other hand, for each $x \in \mathcal{X}$, we have

$$\begin{aligned} (F'n)(x) &= \langle n \cdot \delta_x, Fm \rangle \\ &= \langle (n \cdot \delta_x) \bullet m, F \rangle && (\text{since } Fm \in LUC(\mathcal{X}, \mathcal{G})) \\ &= \langle m \cdot (n \cdot \delta_x), F \rangle && (\text{since } m \in \mathfrak{Z}(\mathcal{X}, \mathcal{G})) \\ &= \langle (m \odot n) \cdot \delta_x, F \rangle \\ &= (F(m \odot n))(x). \end{aligned}$$

Hence, $F(m \odot n)$ is in $LUC(\mathcal{X}, \mathcal{G})$. Moreover, for each $M \in LUC(\mathcal{X}, \mathcal{G})^*$, we have

$$\begin{aligned} \langle (m \odot n) \cdot M, F \rangle &= \langle m \cdot (n \cdot M), F \rangle \\ &= \langle (n \cdot M) \bullet m, F \rangle \\ &= \langle M \bullet n, Fm \rangle \\ &= \langle M, F'n \rangle \\ &= \langle M \bullet (m \odot n), F \rangle. \end{aligned}$$

Therefore, Theorem 4.3 implies that $m \odot n$ is in $\mathfrak{Z}(\mathcal{X}, \mathcal{G})$.

Now, for the proof of the assertion (b), let (m_k) be a sequence in $\mathfrak{Z}(\mathcal{X}, \mathcal{G})$ which tends to $m \in LUC(\mathcal{G})^*$ with respect to the norm topology. Also, suppose that $F \in LUC(\mathcal{X}, \mathcal{G})$ and

$M \in LUC(\mathcal{X}, \mathcal{G})^*$ are given. From Lemma 4.2, we observe that

$$\begin{aligned} \|Fm_k - Fm\|_\infty &= \sup_{x \in \mathcal{X}} |Fm_k(x) - Fm(x)| \\ &= \sup_{x \in \mathcal{X}} |\langle (m_k - m) \cdot \delta_x, F \rangle| \\ &\leq \|m_k - m\| \|F\|_\infty. \end{aligned}$$

This together with the fact that $m_k \in \mathfrak{Z}(\mathcal{X}, \mathcal{G})$ implies that Fm is an element of $LUC(\mathcal{X}, \mathcal{G})$. On the other hand,

$$\langle m \cdot M, F \rangle = \lim_k \langle M \bullet m_k, F \rangle = \lim_k \langle M, Fm_k \rangle = \langle M, Fm \rangle = \langle M \bullet m, F \rangle.$$

We now invoke Theorem 4.3 to conclude that $m \in \mathfrak{Z}(\mathcal{X}, \mathcal{G})$.

Finally, for the proof of the last assertion, let μ be a measure in $M(\mathcal{G})$. Since the measures in $M(\mathcal{G})$ with compact supports are norm dense in $M(\mathcal{G})$, without loss of generality, we may assume that μ has compact support. Let also (M_α) be a bounded net in $LUC(\mathcal{X}, \mathcal{G})^*$ such that $M_\alpha \rightarrow M$ with respect to the weak* topology, for some $M \in LUC(\mathcal{X}, \mathcal{G})^*$. Choose $K > 0$ such that $\|M_\alpha\|, \|M\| \leq K$ for all α . For each $F \in LUC(\mathcal{X}, \mathcal{G})$, α and $s, t \in \mathcal{G}$ we have

$$|M_\alpha F(s) - M_\alpha F(t)| = |\langle M_\alpha, l_s F \rangle - \langle M_\alpha, l_t F \rangle| \leq K \|l_s F - l_t F\|_\infty.$$

Hence, the family of functions $M_\alpha F$ is equicontinuous. Therefore, since $M_\alpha F \rightarrow MF$ pointwisely, the net $M_\alpha F$ convergence uniformly to MF on every compact subset of \mathcal{G} . Thus,

$$|\langle \mu \cdot M_\alpha, F \rangle - \langle \mu \cdot M, F \rangle| = \left| \int_{\mathcal{G}} M_\alpha F d\mu - \int_{\mathcal{G}} MF d\mu \right| \rightarrow 0.$$

It follows that the map $M \mapsto \mu \cdot M$ is weak* continuous on all bounded parts of $LUC(\mathcal{X}, \mathcal{G})^*$. We now invoke Corollary 4.4 to conclude that $\mu \in \mathfrak{Z}(\mathcal{X}, \mathcal{G})$. \blacksquare

The following result gives some necessary and sufficient conditions for the validity of the equality $\mathfrak{Z}(\mathcal{X}, \mathcal{G}) = M(\mathcal{G})$.

Theorem 4.7 *Let \mathcal{G} be a locally compact non-compact group and let \mathcal{X} be a \mathcal{G} -space. Then the following assertions are equivalent.*

- (a) $\mathfrak{Z}(\mathcal{X}, \mathcal{G}) = M(\mathcal{G})$.
- (b) $\mathcal{CLS}(\mathcal{X}, \mathcal{G}) = LUC(\mathcal{G})$.
- (c) *The action of $LUC(\mathcal{G})^*$ on $LUC(\mathcal{X}, \mathcal{G})^*$ is faithful.*

Proof. By Proposition 3.5, we only need to prove (a) \Leftrightarrow (b). In order to prove that (a) implies (b), suppose that $\mathfrak{Z}(\mathcal{X}, \mathcal{G}) = M(\mathcal{G})$. If we had an element f in $LUC(\mathcal{G})$ not belonging to the subspace $\mathcal{CLS}(\mathcal{X}, \mathcal{G})$, then, by the Hahn-Banach Theorem, we would have a functional $m \in LUC(\mathcal{G})^*$ vanishing on the subspace $\mathcal{CLS}(\mathcal{X}, \mathcal{G})$ and such that $\langle m, f \rangle \neq 0$. From this, we deduce that $m \cdot M = 0$ for all $M \in LUC(\mathcal{X}, \mathcal{G})^*$. It follows that $m \in \mathfrak{Z}(\mathcal{X}, \mathcal{G}) = M(\mathcal{G})$. Now, let

$$C_0(\mathcal{G})^\perp = \left\{ n \in LUC(\mathcal{G})^* : \langle n, f \rangle = 0 \text{ for all } f \in C_0(\mathcal{G}) \right\},$$

and pick any $0 \neq m' \in C_0(\mathcal{G})^\perp$ that is right cancellable in $LUC(\mathcal{G})^*$ (such points exists by [7, Theorem 4]). Note that for m' and each $M \in LUC(\mathcal{X}, \mathcal{G})^*$, we have

$$\langle (m \odot m') \cdot M, F \rangle = \langle m \cdot (m' \cdot M), F \rangle = 0$$

for all $F \in LUC(\mathcal{X}, \mathcal{G})$. Therefore, $(m \odot m') \cdot M = 0$ for all $M \in LUC(\mathcal{X}, \mathcal{G})^*$ and this implies that $m \odot m' \in \mathfrak{Z}(\mathcal{X}, \mathcal{G}) = M(\mathcal{G})$. From this, by [10, Lemma 1.1], we deduce that

$$m \odot m' \in M(\mathcal{G}) \cap C_0(\mathcal{G})^\perp = \{0\}.$$

This together with the fact that $0 \neq m'$ is right cancellable, implies that $m = 0$ which is impossible.

To prove implication (b) \Rightarrow (a), suppose that $\mathcal{CLS}(\mathcal{X}, \mathcal{G}) = LUC(\mathcal{G})$. By part (c) of Proposition 4.6, the inclusion $M(\mathcal{G}) \subseteq \mathfrak{Z}(\mathcal{X}, \mathcal{G})$ being always true, it will be enough to prove the reverse inclusion. To this end, suppose that $m \in \mathfrak{Z}(\mathcal{X}, \mathcal{G})$ and (n_α) is a bounded net in $LUC(\mathcal{G})^*$ such that $n_\alpha \longrightarrow n$ in $LUC(\mathcal{G})^*$ with respect to the weak*-topology. Hence, for every $M \in LUC(\mathcal{X}, \mathcal{G})^*$ and $F \in LUC(\mathcal{X}, \mathcal{G})$, we have

$$\begin{aligned} \langle m \odot n_\alpha, MF \rangle &= \langle (m \odot n_\alpha) \cdot M, F \rangle \\ &= \langle m \cdot (n_\alpha \cdot M), F \rangle \\ &\longrightarrow \langle m \cdot (n \cdot M), F \rangle \\ &= \langle m \odot n, MF \rangle; \end{aligned}$$

This is because of, the net $(n_\alpha \cdot M)$ is a net in $LUC(\mathcal{X}, \mathcal{G})^*$ which converges to nM in the weak* topology of $LUC(\mathcal{X}, \mathcal{G})^*$. We now invoke the equality $\mathcal{CLS}(\mathcal{X}, \mathcal{G}) = LUC(\mathcal{G})$ to conclude that

$$\langle m \odot n_\alpha, f \rangle \longrightarrow \langle m \odot n, f \rangle,$$

for all $f \in LUC(\mathcal{G})$. From this, by [14, Theorem 4.1 and Lemma 2.2], we deduce that m belongs to $\mathfrak{Z}(\mathcal{G}) = M(\mathcal{G})$. Therefore, $\mathfrak{Z}(\mathcal{X}, \mathcal{G}) \subseteq M(\mathcal{G})$. Hence, we have the equality $\mathfrak{Z}(\mathcal{X}, \mathcal{G}) = M(\mathcal{G})$. ■

We conclude this section with the following result which is a consequence of Theorem 3.1, part (b) of Proposition 3.5 and Theorem 4.7.

Corollary 4.8 *Let \mathcal{G} be a locally compact non-compact group and \mathcal{X} be a \mathcal{G} -space. If \mathcal{G} does not act effectively on \mathcal{X} , then $M(\mathcal{G})$ is properly contained in $\mathfrak{Z}(\mathcal{X}, \mathcal{G})$.*

5 Examples

This section is devoted to examples with two different purposes. First, illustrating the results presented in this paper for certain \mathcal{G} -spaces \mathcal{X} . Second, to give some example for which

- $\mathfrak{Z}(\mathcal{X}, \mathcal{G}) = M(\mathcal{G})$;
- $M(\mathcal{G}) \subsetneq \mathfrak{Z}(\mathcal{X}, \mathcal{G}) = LUC(\mathcal{G})^*$;
- $\mathfrak{Z}(\mathcal{X}, \mathcal{G})$ is neither $M(\mathcal{G})$ nor $LUC(\mathcal{G})^*$.

To this end, we commence this section with the following example which characterize the subalgebra $\mathfrak{Z}(\mathcal{X}, \mathcal{G})$ of $LUC(\mathcal{G})^*$ for certain \mathcal{G} -spaces \mathcal{X} .

Example 5.1 Let \mathcal{G} be a locally compact group and \mathcal{X} be a \mathcal{G} -space.

(a) If $\mathcal{X} = \mathcal{G}$, then $M(\mathcal{G}) = \mathfrak{Z}(\mathcal{X}, \mathcal{G})$.

(b) If \mathcal{G} is compact, then

$$M(\mathcal{G}) \subseteq \mathfrak{Z}(\mathcal{X}, \mathcal{G}) \subseteq LUC(\mathcal{G})^* = M(\mathcal{G})$$

which shows that $\mathfrak{Z}(\mathcal{X}, \mathcal{G}) = M(\mathcal{G})$.

(c) If \mathcal{X} is a finite discrete space, then $\mathfrak{Z}(\mathcal{X}, \mathcal{G}) = LUC(\mathcal{G})^*$; Indeed, the vector space $LUC(\mathcal{X}, \mathcal{G})$, as a subspace of $C_b(\mathcal{X})$, is a finite dimensional vector space, and so all locally convex topologies on finite dimensional space $LUC(\mathcal{X}, \mathcal{G})^*$ are coincide. In particular, for each $m \in LUC(\mathcal{G})^*$ the linear map $M \mapsto m \cdot M$ is continuous with respect to the norm topology of $LUC(\mathcal{X}, \mathcal{G})^*$, and also it is continuous with respect to the weak* topology. Therefore, $\mathfrak{Z}(\mathcal{X}, \mathcal{G}) = LUC(\mathcal{G})^*$. Specially, if \mathcal{G} is non-compact, then $M(\mathcal{G}) \subsetneq \mathfrak{Z}(\mathcal{X}, \mathcal{G}) = LUC(\mathcal{G})^*$.

It is worthwhile to mention that when \mathcal{G} is a locally compact group and \mathcal{H} is a closed subgroup of \mathcal{G} , then the space \mathcal{G}/\mathcal{H} consisting of all left cosets of \mathcal{H} in \mathcal{G} is a locally compact Hausdorff topological space on which \mathcal{G} acts from the left by

$$\mathcal{G} \times \mathcal{G}/\mathcal{H} \rightarrow \mathcal{G}/\mathcal{H}; (s, t\mathcal{H}) \mapsto (st)\mathcal{H}.$$

It has been shown that if \mathcal{G} is σ -compact, then every transitive \mathcal{G} -space is homeomorphic to the quotient space \mathcal{G}/\mathcal{H} for some closed subgroup \mathcal{H} , see [8, Subsection 2.6].

In what follows, the notation $\mathfrak{C}_{\mathcal{G}}(\mathcal{H})$ is used to denote the centralizer of the subgroup \mathcal{H} in \mathcal{G} ; That is,

$$\mathfrak{C}_{\mathcal{G}}(\mathcal{H}) = \left\{ s \in \mathcal{G} : sh = hs, \text{ for all } h \in \mathcal{H} \right\}.$$

Example 5.2 Let \mathcal{G} be a locally compact, non-compact group and let \mathcal{H} and \mathcal{K} be two non-trivial closed normal subgroups of \mathcal{G} .

(a) If $\mathcal{X} = \mathcal{G}/\mathcal{H}$, then, by Corollary 4.8, $M(\mathcal{G}) \subsetneq \mathfrak{Z}(\mathcal{X}, \mathcal{G})$; This is because of, in this case

$$\mathfrak{N}(\mathcal{X}, \mathcal{G}) = \left\{ s \in \mathcal{G} : st\mathcal{H} = t\mathcal{H}, t \in \mathcal{G} \right\} = \mathcal{H}.$$

In particular, if the index of \mathcal{H} in \mathcal{G} is finite, then, by Proposition 4.5, we have $\mathfrak{Z}(\mathcal{X}, \mathcal{G}) = LUC(\mathcal{G})^*$.

(b) If $\mathcal{X} = \mathcal{K}$, then, obviously, \mathcal{G} acts on \mathcal{X} by conjugation and makes \mathcal{X} into a \mathcal{G} -space; That is,

$$\mathcal{G} \times \mathcal{X} \rightarrow \mathcal{X}; (s, x) \mapsto x^s := sxs^{-1}.$$

In this case, $\mathfrak{N}(\mathcal{X}, \mathcal{G}) = \mathfrak{C}_{\mathcal{G}}(\mathcal{K})$. Hence, we can say if either $\mathfrak{C}_{\mathcal{G}}(\mathcal{G})$ or $\mathfrak{C}_{\mathcal{G}}(\mathcal{K})$ is non-trivial (for example, if \mathcal{K} is abelian), then we have $M(\mathcal{G}) \subsetneq \mathfrak{Z}(\mathcal{X}, \mathcal{G})$.

Now, we consider the following special case of the previous example which illustrates Proposition 4.5 and gives an example of \mathcal{G} and \mathcal{X} such that $M(\mathcal{G}) \subsetneq \mathfrak{Z}(\mathcal{X}, \mathcal{G}) = LUC(\mathcal{G})^*$.

Example 5.3 Let \mathbb{Z} be the discrete group of integer numbers and let, for each $n \in \mathbb{Z}^+$, $n\mathbb{Z}$ be the subgroup generated by n consists of all integer multiples of n . If $\mathcal{X}_n = \mathbb{Z}/n\mathbb{Z}$, then Corollary 4.8 together with the fact that $\mathfrak{N}(\mathcal{X}_n, \mathbb{Z}) = n\mathbb{Z}$ implies that $M(\mathbb{Z}) \subsetneq \mathfrak{Z}(\mathcal{X}_n, \mathbb{Z})$. On the other hand, since the index of $n\mathbb{Z}$ in \mathbb{Z} is n , by Proposition 4.5, we have $\mathfrak{Z}(\mathcal{X}_n, \mathbb{Z}) = LUC(\mathbb{Z})^*$.

The following example illustrates Theorem 4.7.

Example 5.4 Suppose that \mathcal{Y} is an arbitrary locally compact Hausdorff space and \mathcal{G} is a locally compact group. Then $\mathcal{X} = \mathcal{G} \times \mathcal{Y}$, equipped with the product topology and the action defined by

$$\mathcal{G} \times \mathcal{X} \rightarrow \mathcal{X}; s \cdot (t, y) = (st, y),$$

is a \mathcal{G} -space for which $\mathfrak{Z}(\mathcal{X}, \mathcal{G}) = M(\mathcal{G})$; Indeed, by part (b) of Example 5.1, we only need to show that the equality $\mathfrak{Z}(\mathcal{X}, \mathcal{G}) = M(\mathcal{G})$ is valid when \mathcal{G} is non-compact. To this end, first note that, if $f \in LUC(\mathcal{G})$ and $F \in C_b(\mathcal{Y})$, then the function

$$(f \otimes F) : \mathcal{X} \rightarrow \mathbb{C}; (t, y) \mapsto f(t)F(y),$$

is a function in $LUC(\mathcal{X}, \mathcal{G})$; This is because of,

$$l_s(f \otimes F) = ({}_sf) \otimes F,$$

for all $s \in \mathcal{G}$. In particular, if $1_{\mathcal{Y}}$ denotes the constant function 1 on \mathcal{Y} , then, for each $f \in LUC(\mathcal{G})$, the function $f \otimes 1_{\mathcal{Y}}$ is a function in $LUC(\mathcal{X}, \mathcal{G})$ and therefore we can consider $LUC(\mathcal{G})$ as a closed subspace of $LUC(\mathcal{X}, \mathcal{G})$ via the inclusion mapping

$$\iota : LUC(\mathcal{G}) \rightarrow LUC(\mathcal{X}, \mathcal{G}); f \mapsto f \otimes 1_{\mathcal{Y}}.$$

Then $\iota^* : LUC(\mathcal{X}, \mathcal{G})^* \rightarrow LUC(\mathcal{G})^*$ is the restriction mapping and hence norm decreasing and onto by Hahn-Banach Theorem. Moreover, if $m \in LUC(\mathcal{G})^*$ and $M \in LUC(\mathcal{X}, \mathcal{G})^*$ is chosen so that $\iota^*(M) = m$, then

$$\begin{aligned} (m \diamond f)(s) &= \langle m, {}_sf \rangle \\ &= \langle \iota^*(M), {}_sf \rangle \\ &= \langle M, ({}_sf) \otimes 1_{\mathcal{Y}} \rangle \\ &= \langle M, l_s(f \otimes 1_{\mathcal{Y}}) \rangle \\ &= (M(f \otimes 1_{\mathcal{Y}}))(s), \end{aligned}$$

for all $f \in LUC(\mathcal{G})$ and $s \in \mathcal{G}$. Hence, we have

$$\{m \diamond f : m \in LUC(\mathcal{G})^*, f \in LUC(\mathcal{G})\} \subseteq \{MF : M \in LUC(\mathcal{X}, \mathcal{G})^*, F \in LUC(\mathcal{X}, \mathcal{G})\},$$

which shows that $\mathcal{CLS}(\mathcal{X}, \mathcal{G}) = LUC(\mathcal{G})$. We now invoke Theorem 4.7 to conclude that $\mathfrak{Z}(\mathcal{X}, \mathcal{G}) = M(\mathcal{G})$.

The proof of Example 5.6 below, which gives a \mathcal{G} -space \mathcal{X} such that $\mathfrak{N}(\mathcal{X}, \mathcal{G})$ is neither $M(\mathcal{G})$ nor $LUC(\mathcal{G})^*$, relies on the following example.

Example 5.5 Let \mathcal{G}_1 and \mathcal{G}_2 be two locally compact groups. Then we can consider $LUC(\mathcal{G}_1)$ as a closed subspace of $LUC(\mathcal{G}_1 \times \mathcal{G}_2)$ via the inclusion mapping

$$\iota_{\mathcal{G}_1} : LUC(\mathcal{G}_1) \rightarrow LUC(\mathcal{G}_1 \times \mathcal{G}_2); f \mapsto f \otimes 1_{\mathcal{G}_2},$$

where $1_{\mathcal{G}_2}$ denotes the constant function 1 on \mathcal{G}_2 . Suppose also that \mathcal{X}_1 is a \mathcal{G}_1 -space and \mathcal{X}_2 is a \mathcal{G}_2 -space. Then $\mathcal{X}_1 \times \mathcal{X}_2$, with product topology and coordinatewise left module action, is a $\mathcal{G}_1 \times \mathcal{G}_2$ -space for which

$$\iota_{\mathcal{G}_1}^* \left(\mathfrak{Z}(\mathcal{X}_1 \times \mathcal{X}_2, \mathcal{G}_1 \times \mathcal{G}_2) \right) \subseteq \mathfrak{Z}(\mathcal{X}_1, \mathcal{G}_1).$$

In particular, since the map $\iota_{\mathcal{G}_1}^*$ is onto, if

$$\mathfrak{Z}(\mathcal{X}_1 \times \mathcal{X}_2, \mathcal{G}_1 \times \mathcal{G}_2) = LUC(\mathcal{G}_1 \times \mathcal{G}_2)^*,$$

then $\mathfrak{Z}(\mathcal{X}_1, \mathcal{G}_1) = LUC(\mathcal{G}_1)^*$; Indeed, if $m \in \mathfrak{Z}(\mathcal{X}_1 \times \mathcal{X}_2, \mathcal{G}_1 \times \mathcal{G}_2)$, $F_1 \in LUC(\mathcal{X}_1, \mathcal{G}_1)$ and $1_{\mathcal{X}_2}$ denotes the constant function on \mathcal{X}_2 with value 1, then $F_1 \otimes 1_{\mathcal{X}_2} \in LUC(\mathcal{X}_1 \times \mathcal{X}_2, \mathcal{G}_1 \times \mathcal{G}_2)$ and we have

$$\begin{aligned} \left((F_1 \otimes 1_{\mathcal{X}_2})m \right)(x_1, x_2) &= \langle m, r_{(x_1, x_2)}(F_1 \otimes 1_{\mathcal{X}_2}) \rangle \\ &= \langle m, (r_{x_1} F_1) \otimes 1_{\mathcal{X}_2} \rangle \\ &= \langle \iota_{\mathcal{G}_1}^*(m), r_{x_1} F_1 \rangle \\ &= \left(F_1 \iota_{\mathcal{G}_1}^*(m) \right)(x_1), \end{aligned}$$

for all $x_1 \in \mathcal{X}_1$ and $x_2 \in \mathcal{X}_2$ and therefore

$$(F_1 \otimes 1_{\mathcal{X}_2})m = (F_1 \iota_{\mathcal{G}_1}^*(m)) \otimes 1_{\mathcal{X}_2}.$$

From this, by Theorem 4.3, we have

$$(F_1 \iota_{\mathcal{G}_1}^*(m)) \otimes 1_{\mathcal{X}_2} \in LUC(\mathcal{X}_1 \times \mathcal{X}_2, \mathcal{G}_1 \times \mathcal{G}_2).$$

It follows that $F_1 \iota_{\mathcal{G}_1}^*(m) \in LUC(\mathcal{X}_1, \mathcal{G}_1)$ for all $F_1 \in LUC(\mathcal{X}_1, \mathcal{G}_1)$. Moreover, if

$$\iota_{\mathcal{X}_1} : LUC(\mathcal{X}_1, \mathcal{G}_1) \rightarrow LUC(\mathcal{X}_1 \times \mathcal{X}_2, \mathcal{G}_1 \times \mathcal{G}_2); F_1 \mapsto F_1 \otimes 1_{\mathcal{X}_2},$$

then it is not hard to check that

$$M(F_1 \otimes 1_{\mathcal{X}_2}) = (\iota_{\mathcal{X}_1}^*(M)F_1) \otimes 1_{\mathcal{G}_2},$$

for all $F_1 \in LUC(\mathcal{X}_1, \mathcal{G}_1)$, where $1_{\mathcal{G}_2}$ denotes the constant function one on \mathcal{G}_2 . It follows that

$$\iota_{\mathcal{G}_1}^*(m) \cdot \iota_{\mathcal{X}_1}^*(M) = \iota_{\mathcal{X}_1}^*(M) \bullet \iota_{\mathcal{G}_1}^*(m),$$

for all $M \in LUC(\mathcal{X}_1 \times \mathcal{X}_2, \mathcal{G}_1 \times \mathcal{G}_2)^*$; In details, if $F_1 \in LUC(\mathcal{X}_1, \mathcal{G}_1)$, then, we can write

$$\begin{aligned} \langle \iota_{\mathcal{G}_1}^*(m) \cdot \iota_{\mathcal{X}_1}^*(M), F_1 \rangle &= \langle \iota_{\mathcal{G}_1}^*(m), \iota_{\mathcal{X}_1}^*(M)F_1 \rangle \\ &= \langle m, (\iota_{\mathcal{X}_1}^*(M)F_1) \otimes 1_{\mathcal{G}_2} \rangle \\ &= \langle m, M(F_1 \otimes 1_{\mathcal{X}_2}) \rangle \end{aligned}$$

$$\begin{aligned}
&= \langle m \cdot M, F_1 \otimes 1_{\mathcal{X}_2} \rangle \\
&= \langle M \bullet m, F_1 \otimes 1_{\mathcal{X}_2} \rangle \quad \left(\text{since } m \in \mathfrak{Z}(\mathcal{X}_1 \times \mathcal{X}_2, \mathcal{G}_1 \times \mathcal{G}_2) \right) \\
&= \langle M, (F_1 \otimes 1_{\mathcal{X}_2})m \rangle \\
&= \langle M, (F_1 \iota_{\mathcal{G}_1}^*(m)) \otimes 1_{\mathcal{X}_2} \rangle \\
&= \langle \iota_{\mathcal{X}_1}^*(M), F_1 \iota_{\mathcal{G}_1}^*(m) \rangle \\
&= \langle \iota_{\mathcal{X}_1}^*(M) \bullet \iota_{\mathcal{G}_1}^*(m), F_1 \rangle. \quad \left(\text{since } F_1 \iota_{\mathcal{G}_1}^*(m) \in LUC(\mathcal{X}_1, \mathcal{G}_1) \right)
\end{aligned}$$

Hence, since the map $\iota_{\mathcal{X}_1}^*$ is onto, we have

$$\iota_{\mathcal{G}_1}^*(m) \cdot M_1 = M_1 \bullet \iota_{\mathcal{G}_1}^*(m),$$

for all $M_1 \in LUC(\mathcal{X}_1, \mathcal{G}_1)^*$. We now invoke Theorem 4.3 to conclude that $\iota_{\mathcal{G}_1}^*(m) \in \mathfrak{Z}(\mathcal{X}_1, \mathcal{G}_1)$.

Example 5.6 Let Q_8 be the quaternion group, the subgroup of the general linear group $GL(2, \mathbb{C})$ generated by the matrices

$$\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{i} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \mathbf{j} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathbf{k} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

which can be also presented by

$$Q_8 = \langle a, b : a^4 = 1, a^2 = b^2, b^{-1}ab = a^{-1} \rangle,$$

where $a = \mathbf{i}$ and $b = \mathbf{j}$ for instance. Take $\mathcal{K} = \langle a \rangle$, which is a normal subgroup of Q_8 . If we set $\mathcal{G} = Q_8 \times \mathbb{F}_2$, the direct product of Q_8 and the free group on a two-element set, then $\mathcal{X} = \mathcal{K} \times \mathbb{F}_2$ is a closed normal subgroup of discrete topological group \mathcal{G} and so, \mathcal{G} acts on \mathcal{X} by

$$\mathcal{G} \times \mathcal{X} \rightarrow \mathcal{X} : (s, x) \mapsto sxs^{-1}.$$

Since the algebraic centre $\mathfrak{C}_{\mathcal{G}}(\mathcal{G})$ contains $\mathfrak{C}_{Q_8}(Q_8) = \{\mathbf{i}^2, \mathbf{1}\}$, by part (b) of Example 5.2, we have $M(\mathcal{G}) \subsetneq \mathfrak{Z}(\mathcal{X}, \mathcal{G})$. On the other hand, if $\mathfrak{Z}(\mathcal{X}, \mathcal{G}) = LUC(\mathcal{G})^*$, then by Example 5.5, we would have $\mathfrak{Z}(\mathbb{F}_2, \mathbb{F}_2) = LUC(\mathbb{F}_2)^*$, which is impossible; This is because of, since \mathbb{F}_2 is not compact,

$$\mathfrak{Z}(\mathbb{F}_2, \mathbb{F}_2) = \mathfrak{Z}(\mathbb{F}_2) \subsetneq LUC(\mathbb{F}_2)^*.$$

Hence, for \mathcal{G} -space \mathcal{X} , we have $M(\mathcal{G}) \subsetneq \mathfrak{Z}(\mathcal{X}, \mathcal{G}) \subsetneq LUC(\mathcal{G})^*$.

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